HW 2 (today): 2.2 & slope fields
HW 3 (Wed): 2.1 Integrating Factors
2.3 applications

2.1: Integrating Factors
(Linear 1st Order Diff. Eqns)

Recall: We have been solving 1st order equations such as:

$$\frac{dy}{dt} = G(t, y)$$

We call the equation ***linear*** if $G(t, y)$ is a linear function of *y*, that is:

$$G(t, y) = m(t)y + b(t)$$

Otherwise, we say it is **non-linear.**

Many important applications we have seen are linear (populations, bank accounts, air resistance, temperature, etc…), so this is an important special case.

Given a 1st order linear ODE, we like to re-arrange it into the form:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Examples:

1. $\frac{dy}{dt} + ty = t^3$

Give $p(t)$ and $g(t)$.

2. $x\frac{dy}{dx} = \sin(x) - 2y$

Give $p(x)$ and $g(x)$.

Big Observation 1

The form

$$\frac{dy}{dt} + p(t)y$$

looks sort of like the *product rule.*

Recall: Here is the product rule

$$\frac{d}{dt}(f(t)y) = f(t)\frac{dy}{dt} + f'(t)y$$

Example:

$$\frac{dy}{dt} + 2y = \frac{t}{e^{2t}}$$

so $p(t) = 2$ and $g(t) = \frac{t}{e^{2t}}$

What happens if you multiply both sides by e^{2t}?

You get

$$e^{2t}\frac{dy}{dt} + 2e^{2t}y = t$$

which is

$$\frac{d}{dt}(e^{2t}y) = t$$

Integrating both sides gives

$$e^{2t}y = \frac{1}{2}t^2 + C$$

so

$$y = \frac{\frac{1}{2}t^2 + C}{e^{2t}}$$

In this example, we call $\mu(t) = e^{2t}$, the *integrating factor*, which is a function we multiply by so we can reverse the product rule.

Okay, sort of cool, but we were lucky this time, how can we make this work in a more general way?

We need another big observation.

Big Observation 2

If $F(t)$ is any antiderivative of $p(t)$

$$F(t) = \int p(t)dt$$

then

$$\frac{d}{dt}\left(e^{F(t)}y\right) = e^{F(t)}\frac{dy}{dt} + p(t)e^{F(t)}y$$
$$= e^{F(t)}\left(\frac{dy}{dt} + p(t)y\right).$$

So if we multiply by

$$\mu(t) = e^{\int p(t)dt}$$

then we create a situation where we can reverse the product rule.

Integrating Factor Method

Given a <u>linear</u>, 1st order ODE

$$\frac{dy}{dt} = f(t, y)$$

Step 0: Put in form

$$\frac{dy}{dt} + p(t)y = g(t)$$

Step 1: Find $F(t) = \int p(t)dt$

& write/simplify $\mu(t) = e^{\int p(t)dt}$

Step 2: Multiply BOTH sides by $\mu(t)$.
& re-write LHS as product rule.

Step 3: Integrate both sides, and simplify.

Example: $\frac{dy}{dt} = \frac{\cos(t)}{t^2} - \frac{2y}{t}$

Step 0:

$$\frac{dy}{dt} + \frac{2}{t}y = \frac{\cos(t)}{t^2}$$

Step 1: $F(t) = \int \frac{2}{t}dt = 2\ln(t) + C$

$$\mu(t) = e^{2\ln(t)} = e^{\ln(t^2)} = t^2$$

Step 2: $t^2\frac{dy}{dt} + 2ty = \cos(t)$

$$\frac{d}{dt}(t^2 y) = \cos(t)$$

Step 3: $t^2 y = \sin(t) + C$

$$y = \frac{\sin(t) + C}{t^2}$$

Example:

$$3\frac{dy}{dt} - 6ty - 3e^{t^2} = 0$$

Example:

$$\frac{dy}{dt} = t - 3y$$

Two Notes:

- Only for linear 1st order ODEs!

  Aside:
  Again, sometimes a substitution can make it linear.
  Example:
  $$e^y \frac{dy}{dt} - \frac{1}{x} e^y = 3x \quad \text{is not linear}$$
  Using
  $$v = e^y \rightarrow \frac{dv}{dt} = e^y \frac{dy}{dt}$$
  Changes it to
  $$\frac{dv}{dt} - \frac{1}{x} v = 3x \quad \text{which is linear}$$

- If we can't do the integrals we can still write our answer in terms of integrals.

  In which case the convention is to write
  $$\int f(t)dt = \int_0^t f(u)du + C$$
  (so we can solve for C if needed).
  See next page for an example.

Example:

$$\frac{1}{6}\frac{dy}{dt} + t^2 y = \frac{1}{6}$$